A COMBINATORIAL REALIZATION OF THE HEISENBERG ACTION ON THE SPACE OF CONFORMAL BLOCKS

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Abstract. In this article we construct a combinatorial counterpart of the action of a certain Heisenberg group on the space of conformal blocks which was studied by Andersen-Masbaum [1] and Blanchet-Habegger-Masbaum-Vogel [5]. It is a non-abelian analogue of the description of the Heisenberg action on the space of theta functions.

1. Introduction

The space of conformal blocks in conformal field theory has several significant features. One aspect is that it is regarded as a TQFT-module in $(2+1)$-dimensional topological quantum field theory. Our interest in the present paper is an action of a certain Heisenberg group, which was constructed by Blanchet-Habegger-Masbaum-Vogel [5] in the $SU(2)$-case[1]. For each oriented surface, they constructed a TQFT-module[2] based on the theory of the two dimensional cobordism category and topological invariants of three manifolds. Their Heisenberg group is a central extension of the first homology group of the surface with $\mathbb{Z}/2$-coefficient, and they constructed an action of the group on their TQFT-module by making use of a product cobordism. They obtained the character formula and determined the weight decomposition of the action, which is the starting point of what they call *spin-refined TQFT* [4]. On the other hand Andersen-Masbaum [1] obtained the same character formula of the Heisenberg action in the holomorphic setting. In their setting the space of conformal blocks corresponds to the space of holomorphic sections of a holomorphic line bundle over the moduli space of flat $SU(2)$-bundles over a Riemann surface due to the result of Beauville-Laszlo [2]. The Heisenberg group naturally acts on the line bundle, and hence on the space of conformal blocks. Blanchet-Habegger-Masbaum-Vogel showed that their TQFT-module and the space of holomorphic sections have the same dimension. The results of Andersen-Masbaum [1] imply that these two spaces are also isomorphic each other as representations of the Heisenberg group. Note that in [1] and [5], they obtained the character formula but have not obtained the representation matrices. The purpose of this article is to formulate the Heisenberg action in a combinatorial setting and to give the representation matrices explicitly. Blanchet-Habegger-Masbaum-Vogel showed that for each pants decomposition of the surface there exists a canonical basis of the TQFT-module

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[1]Blanchet [3] obtained the corresponding results for the $SU(N)$-case.

[2]Their TQFT-module is not a vector space, but a module over a certain cyclotomic ring.

parameterized by *admissible weights* of the dual trivalent graph of the pants decomposition. On the other hand it is known that we can construct (2+1)-dimensional TQFT, at least partially, starting from the vector space generated by admissible weights of a fixed trivalent graph. See [7] or [9] for example. Recently Yoshida [8] carried out an abelianization program of $SU(2)$ Wess-Zumino-Witten model and constructed an explicit basis of the space of holomorphic sections parameterized by admissible weights.

In this article we describe the Heisenberg action on the space of conformal blocks in terms of a fixed trivalent graph and its admissible weights, and we obtain the representation matrix of the Heisenberg action. Our construction is motivated by Yoshida's construction of the basis. Our approach is rather combinatorial though there might be more direct geometric approach based on Yoshida's construction.

To make our purpose clearer we recall the abelian case. Let C be a closed Riemann surface of genus g and consider the Jacobian $J(C)$ of C. Note that there is the standard identification of $J(C)$ with $Pic(C)$, the moduli space of degree 0 holomorphic line bundles over C. For a positive integer k, let Θ_k be the space of holomorphic sections of the k-th tensor product of a natural holomorphic line bundle over $J(C)$. On the other hand if we fix a symplectic basis of $H_1(C;\mathbb{Z})$ then we have a decomposition into two Lagrangians, $H_1(C;\mathbb{Z}/k) = A_k \oplus B_k$ $(A_k \cong B_k \cong (\mathbb{Z}/k)^g)$. In addition it is well known that there is an explicit description of the basis of Θ_k, that is, theta functions of level k, $\{\theta_\beta \mid \beta \in B_k\}$. Then the Heisenberg group defined as a certain central extension of $H_1(C;\mathbb{Z}/k)$ acts on Θ_k as follows[3];

$$\begin{aligned} \alpha &: \theta_{\beta'} \mapsto e^{2\pi\sqrt{-1}\alpha\cdot\beta'}\theta_{\beta'} \quad (\alpha \in A_k) \\ \beta &: \theta_{\beta'} \mapsto \theta_{\beta+\beta'} \quad (\beta \in B_k). \end{aligned}$$

In particular each θ_α is an eigenvector of the action of A_k, and the action of B_k interchanges the basis of Θ_k.

In the $U(1)$-case for a given symplectic basis of $H_1(C;\mathbb{Z})$ we have an explicit description of a basis of Θ_k and the Heisenberg action on Θ_k. On the other hand as we mentioned before a trivalent graph or a surface with a pants decomposition gives rise to a basis of the space of conformal blocks in the $SU(2)$-case. The purpose of this article is to construct an $SU(2)$-version of the explicit description of the Heisenberg action on the space of conformal blocks in terms of admissible weights of the fixed trivalent graph, as in the $U(1)$-case.

This article is organized as follows. In Section 2, we recall the results of Andersen-Masbaum in [1], and we give the statement of our main results which describe representation matrices of the Heisenberg action. In Section 3 we prepare several combinatorial data and their properties associated with a trivalent graph. We refer to the *factorization property* and the *Verlinde formula* in our combinatorial settings which give the number of admissible weights. We need these formulas to prove Theorem 4.11 in Section 4. We also introduce a decomposition into two Lagrangians, $H_1(C;\mathbb{Z}/2) = A_2 \oplus B_2$, by using the trivalent graph with some additional data. In Section 4 we give representation matrices of the Heisenberg action. We first define actions of A_2 and B_2 on the vector space generated by admissible weights in an explicit combinatorial way. These actions correspond respectively to the actions of A_2 and B_2 on the space of theta functions in the $U(1)$-case. The character formula of this A_2-action coincides with that in [1] and [5]. On the other hand, though

[3]An element c in the center acts on Θ_k by the multiplication of c^k.

there is a B_2-action defined in a naive way, the character of the naive action does not coincide. Hence one needs a modification of the action. Such a modification is given by a certain cocycle. We introduce a geometric/combinatorial condition for cocycles, the *external edge condition* (Definition 4.6). One of our main results (Theorem 4.11) is that the external edge condition is a sufficient condition to reconstruct the Heisenberg action in [1] and [5]. The external edge condition is a natural condition from the view point of the factorization. See also [6]. In Subsection 4.4 we define a Heisenberg group as a central extension of $H_1(C;\mathbb{Z}/2)$ and an action of the group on the vector space generated by admissible weights. Under the external edge condition we show that our Heisenberg action have the same character in [1] and [5] (Theorem 4.16). In Section 5 we give explicit constructions of cocycles satisfying the external edge condition for all planar trivalent graphs in Theorem 5.2 and some non-planar graphs in Example 5.4. The existence of such cocycles for all trivalent graph is shown in [6]. Moreover in [6] we show that there is a finite algorithm to construct such cocycles. The constructions given in Section 5 are examples given by that algorithm.

2. Statement of the main results

First we review the construction of Andersen-Masbaum in [1] briefly. Let R_g be the moduli space of flat $SU(2)$-bundles over a closed Riemann surface C of genus g. We assume that $g \geq 2$. It is well known that R_g has a natural structure of (singular) Kähler manifold of $\dim_{\mathbb{C}} R_g = 3g-3$. There is a natural holomorphic Hermitian line bundle $\mathcal{L} \to R_g$ whose curvature form coincides with the Kähler form on R_g. The space of holomorphic sections $H^0(R_g;\mathcal{L}^{\otimes k})$ is called the space of non-abelian theta functions and it is a realization of the space of conformal blocks. Let $J^{(r)}$ be the subgroup consisting of order r points in $J(C) = Pic(C)$. Note that $J^{(r)}$ is identified with $H^1(C;\mathbb{Z}/r)$. Then $J^{(2)}$ acts on R_g by the tensor product of vector bundles. For each $\alpha \in J^{(2)}$ there is an involutive lift of $\alpha : R_g \to R_g$ to $\mathcal{L}$ since α acts on $Pic(R_g)$ trivially, and such a lift is unique up to sign. If we choose a square root of α, then we can fix the sign as follows. Let $a \in J^{(4)}$ be a square root of α, i.e, $2a = \alpha$. Denote by L_a the flat line bundle corresponding to a. Then the point in R_g represented by the flat $SU(2)$-bundle $L_a \oplus L_{-a}$ is fixed by the action of α. These data determine a lift $\rho_a : \mathcal{L} \to \mathcal{L}$ of α, i.e, ρ_a is the lift of α which acts trivially on the fiber of $\mathcal{L}$ over $L_a \oplus L_{-a}$[4]. Let $\mathcal{G}(J^{(2)};\mathcal{L})$ be the group consisting of the automorphisms of $\mathcal{L}$ covering the action of $J^{(2)}$ on R_g (not necessarily of order 2) and let $\mathcal{E}(J^{(2)};\mathcal{L})$ be the subgroup of $\mathcal{G}(J^{(2)};\mathcal{L})$ generated by involutions $\{\rho_a \mid a \in J^{(4)}\}$. Andersen and Masbaum determined the group structure of $\mathcal{E}(J^{(2)};\mathcal{L})$ by analyzing the fixed point varieties of the action of $J^{(2)}$ on R_g.

Theorem 2.1 (J. E. Andersen and G. Masbaum [1]). *The group structure of $\mathcal{E}(J^{(2)};\mathcal{L})$ is given by*

$$\rho_a\rho_b = \sqrt{-1}^{\omega_4(a,b)}\rho_{a+b} \quad (a, b \in J^{(4)}),$$

[4]By definition and Theorem 2.1, one knows the dependence of lifts on a choice of square roots of α. If a and a' are two square root of α then one has $\rho_a = \sqrt{-1}^{\omega_4(a,a')}\rho_{a'}$.

where ω_4 is the intersection form on $J^{(4)} = H^1(C;\mathbb{Z}/4)$[5].

The above theorem implies that the natural surjection $\mathcal{E}(J^{(2)};\mathcal{L}) \to J^{(2)}$ defines a central extension

$$0 \to \mathbb{Z}/4 \to \mathcal{E}(J^{(2)};\mathcal{L}) \to J^{(2)} \to 1.$$

Remark 2.1. Note that the commutator of $\mathcal{E}(J^{(2)};\mathcal{L})$ is given by the intersection form on $J^{(2)}$ and it is known that $\mathcal{E}(J^{(2)};\mathcal{L})$ can be reduced to a $\mathbb{Z}/2$-central extension of $J^{(2)}$. See [1, p.297]. In Proposition 4.15 we give such a reduction explicitly by using a pants decomposition of the surface.

For each positive integer k, there is a natural action of $\mathcal{E}(J^{(2)};\mathcal{L})$ on $H^0(R_g;\mathcal{L}^{\otimes k})$ and there are involutions $\rho_a^{\otimes k} : H^0(R_g;\mathcal{L}^{\otimes k}) \to H^0(R_g;\mathcal{L}^{\otimes k})$. Andersen-Masbaum obtained the following character formula of these involutions by using the holomorphic Lefschetz formula[6]. Their formula characterizes the representation of $\mathcal{E}(J^{(2)};\mathcal{L})$ on $H^0(R_g;\mathcal{L}^{\otimes k})$ up to isomorphism.

Theorem 2.2 (J. E. Andersen and G. Masbaum [1])**.**

$$Tr(\rho_a; H^0(R_g;\mathcal{L}^{\otimes k})) = \frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1} \qquad (2a = \alpha \neq 0).$$

In [8] Yoshida constructed an explicit basis of $H^0(R_g;\mathcal{L}^{\otimes k})$ by making use of a fixed pants decomposition of the surface. One can associate a trivalent graph to a pants decomposition as follows. The set of vertices of the graph is the set of pairs of pants, and the set of edges is the set of boundary curves. Two vertices v and v' are connected by an edge f if and only if the corresponding pairs of pants P and P' have a common boundary curve e corresponding to f. Conversely for each trivalent graph we can construct a surface with a corresponding pants decomposition, for example, as the boundary of a regular neighborhood of the trivalent graph embedded in $\mathbb{R}^3$. See Fig. 1.

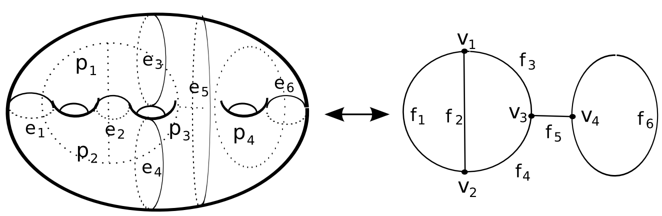

FIGURE 1. A pants decomposition and its dual trivalent graph.

In Subsection 3.2 we define four lattices Λ_0, Λ_0^*, Λ and Λ^* associated with a trivalent graph, and we construct two abelian groups $\Lambda_0/2\Lambda$ and $\Lambda^*/2\Lambda_0^*$ from them. In Proposition 3.5 we show that these abelian groups give a decomposition into two Lagrangians, $H_1(C;\mathbb{Z}/2) \cong \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$.

[5]In fact their description is given by the *order 4 Weil pairing* over $J^{(4)}$ which is equal to the mod 4 symplectic form up to sign.

[6]The trace of the identity is given by the Verlinde formula and an element c in the center acts by the multiplication of c^k

Following these facts our main results can be stated as follows. (The precise statement is given in Proposition 4.15, Theorem 4.16 and Theorem 5.2.)

Main results.

(0) For each trivalent graph Γ, we can construct a $\mathbb{Z}/2$-central extension $\mathcal{E}'(\Gamma)$ of $H_1(C;\mathbb{Z}/2)$, and it is a reduction of the $\mathbb{Z}/4$-central extension $\mathcal{E}(J^{(2)};\mathcal{L})$ to a $\mathbb{Z}/2$-central extension.

(1) For each *twisted 1-cocycle* of the group $\Lambda^*/2\Lambda_0^*$ with values in a certain $\Lambda^*/2\Lambda_0^*$-module, we can construct an action of $\mathcal{E}'(\Gamma)$ on the vector space generated by admissible weights in an explicit combinatorial way, and if the cocycle satisfies the *external edge condition* (Definition 4.6), then the action of induced $\mathbb{Z}/4$-central extension is isomorphic to the action of $\mathcal{E}(J^{(2)};\mathcal{L})$ on the space of holomorphic sections $H^0(R_g;\mathcal{L}^{\otimes k})$.

We also see that the cohomology class defined by a twisted 1-cocycle satisfying the external edge condition is unique. Our main results give a combinatorial reconstruction of the Heisenberg action on the space of conformal blocks. Moreover in [6] we shall show the existence of the cocycles satisfying the external edge condition and have a finite algorithm to construct such cocycles. But we have had no canonical constructions yet. Comparing with the results in [1], [5] and [8], there might be a canonical construction for all trivalent graphs. It is a future problem to find such a canonical construction by comparing our construction with more geometric ones in [1], [5] and [8].

3. Trivalent graph and associated combinatorial data

In this section we recall the definition of *admissible weights* associated with a trivalent graph, and we refer the *factorization property* and the *Verlinde formula*. Also we define four lattices associated with the trivalent graphs, which give a decomposition of the first homology group of the surface into two Lagrangians.

3.1. Admissible weights, factorization property and Verlinde formula.

Our main results concern with trivalent graphs, which correspond to closed surfaces. We also use unitrivalent graphs in the proof of some of our results. In this subsection we make several preparations to treat such graphs.

Let $\Gamma_{g,n} = \{f_l, v_i, w_m \mid l = 1,\cdots,3g-3+2n,\ i = 1,\cdots,2g-2+n,\ m = 1,\cdots,n\}$ be a unitrivalent graph with $3g-3+2n$ edges $\{f_l\}$, $2g-2+n$ trivalent vertices $\{v_i\}$ and n univalent vertices $\{w_m\}$. We assume that $f_1,\cdots,f_n$ have univalent vertices. See Fig.2 for example. Fix n half integers $j'_1,\cdots,j'_n \in \{0,\frac{1}{2},\cdots,\frac{k}{2}\}$.

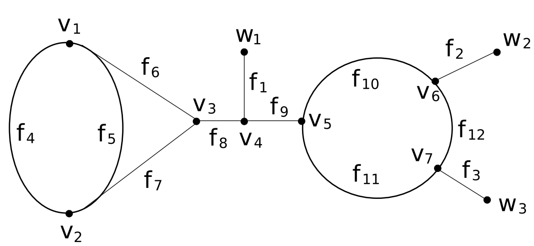

Figure 2. An example of $\Gamma_{g,n}$ for g=3, n=3.

Definition 3.1 (Admissible weights)**.** A labeling of the set of edges $j = (j_l) : \{f_l\} \to \{0, \frac{1}{2}, \cdots, \frac{k}{2}\}$ is an *admissible weight of level* k for $(\Gamma_{g,n}; j'_1, \cdots, j'_n)$ if $j(f_l) = j'_l$ for $l = 1, \cdots n$ and the following condition, which is called the *quantum Clebsch-Gordan condition of level* k, is satisfied for each trivalent vertex v_i with three edges f_{i_1}, f_{i_2} and f_{i_3};

$$\begin{cases} j_{i_1} + j_{i_2} + j_{i_3} \in \mathbb{Z} \\ |j_{i_1} - j_{i_2}| \leq j_{i_3} \leq j_{i_1} + j_{i_2} \\ j_{i_1} + j_{i_2} + j_{i_3} \leq k. \end{cases}$$

Here we put $j_l := j(f_l)$. If a trivalent vertex v_i has only two edges f_{i_1} and $f_{i_2} = f_{i_3}$ then we interpret these conditions as a corresponding condition with $j_{i_2} = j_{i_3}$. Denote the set of all admissible weights for $(\Gamma_{g,n}; j'_1, \cdots, j'_n)$ by $QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n)$. Note that if $j'_1 + \cdots + j'_m \notin \mathbb{Z}$ then $QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n) = \emptyset$. For a trivalent graph (i.e $n = 0$) we denote the set of all admissible weights by $QCG_k(\Gamma_{g,0})$.

It is known that the number of elements in $QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n)$ gives the dimension of the space of conformal block for a Riemann surface with n marked points in the $SU(2)$-case. It is known that the space of conformal blocks has the *factorization property* with respect to degeneration of Riemann surfaces. An aspect of the factorization property is the following combinatorial formulas with respect to cut-and-paste of graphs.

Theorem 3.2 (Factorization property)**.** *For a non-separating edge $f_l \in \Gamma_{g,n}$ consider the graph $\Gamma_{g-1,n+2}$ obtained by cutting $\Gamma_{g,n}$ at f_l (Fig. 3). Then we have the following equality;*

$$^\#QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n) = \sum_{2j=0}^{k} {}^\#QCG_k(\Gamma_{g-1,n+2}; j'_1, \cdots, j'_n, j, j).$$

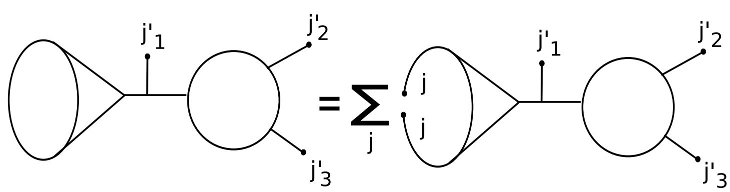

FIGURE 3.

For a separating edge $f_l \in \Gamma_{g,n}$ consider the graph Γ_{g_1,n_1} and Γ_{g_2,n_2} ($g_1 + g_2 = g, n_1 + n_2 = n + 2$) obtained by cutting $\Gamma_{g,n}$ at f_l (Fig. 4). Then we have the following equality;

$$^\#QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n) = \sum_{2j=0}^{k} {}^\#QCG_k(\Gamma_{g_1,n_1}; j'_1, \cdots, j'_{n_1}, j) {}^\#QCG_k(\Gamma_{g_2,n_2}; j, j'_{n_1+1}, \cdots, j'_n).$$

By the factorization property and so-called the *fusion rule*, the number of elements in $QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n)$ is given by the following *Verlinde formula*. (We use the following closed formula to prove Theorem 4.11.)

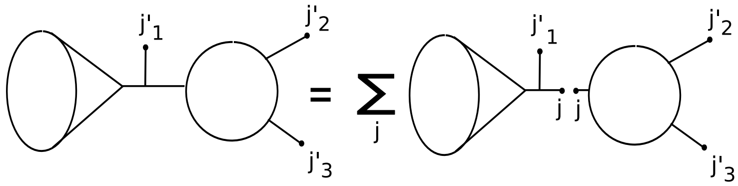

FIGURE 4.

Theorem 3.3 (Verlinde formula)**.**

$$^{\#}QCG_k(\Gamma_{g,n}; j'_1, \cdots, j'_n) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{l=1}^{k+1} \frac{\prod_{m=1}^{n} \left(\sin \frac{2j'_m+1}{k+2} l\pi\right)}{\left(\sin \frac{l\pi}{k+2}\right)^{2g-2+n}}$$

For a trivalent graph $\Gamma = \Gamma_{g,0}$ let $\mathbb{C}(\Gamma; k)$ be the vector space generated by the basis parameterized by the finite set $QCG_k(\Gamma)$. Denote the base corresponding to $j \in QCG_k(\Gamma)$ by $|j\rangle \in \mathbb{C}(\Gamma; k)$. As we mentioned in Introduction one can construct a (2+1)-dimensional TQFT, at least partially, from the vector space $\mathbb{C}(\Gamma; k)$. See [7] or [9] for example [7]. From this point of view the purpose of this article is to construct the action of a Heisenberg group on $\mathbb{C}(\Gamma; k)$ in an explicit combinatorial way and to have representation matrices.

3.2. **Four lattices.** Now we fix a trivalent graph and prepare four lattices which we need later. For $g \geq 2$, let $\Gamma = \{f_l, v_i \mid l = 1, \cdots, 3g-3,\ i = 1, \cdots, 2g-2\}$ be a trivalent graph with $3g-3$ edges $\{f_l\}$ and $2g-2$ vertices $\{v_i\}$. Fix $3g-3$ letters $\{e_l \mid l = 1, \cdots, 3g-3\}$. Let $\Lambda_0 := \mathbb{Z}\langle\{e_l\}\rangle$ (resp. $\Lambda_0^* := \mathbb{Z}\langle\{f_l\}\rangle$) be the standard lattice in $\mathbb{R}\langle\{e_l\}\rangle (\cong \mathbb{R}^{3g-3})$ (resp. $\mathbb{R}\langle\{f_l\}\rangle$), the $\mathbb{R}$-vector space generated by $\{e_l\}$ (resp. $\{f_l\}$).

Definition 3.4. Let Λ be the lattice in $\mathbb{R}\langle\{e_l\}\rangle$ generated by vectors $\{E_1^i, E_2^i, E_3^i \mid i = 1, \cdots, 2g-2\}$, where for a vertex v_i with three edges f_{i_1},f_{i_2} and f_{i_3}, we put

$$\begin{aligned} E_1^i &:= \frac{1}{2}\left(-e_{i_1} + e_{i_2} + e_{i_3}\right) \\ E_2^i &:= \frac{1}{2}\left(e_{i_1} - e_{i_2} + e_{i_3}\right) \\ E_3^i &:= \frac{1}{2}\left(e_{i_1} + e_{i_2} - e_{i_3}\right). \end{aligned}$$

If a vertex v_i has only two edges f_{i_1} and $f_{i_2} = f_{i_3}$ then we interpret as

$$\begin{aligned} E_1^i &:= -\frac{1}{2}e_{i_1} + e_{i_2} \\ E_2^i &= E_3^i := \frac{1}{2}e_{i_1}. \end{aligned}$$

Consider the standard symplectic structure on the vector space $\mathbb{R}\langle\{e_l\}\rangle \oplus \mathbb{R}\langle\{f_l\}\rangle$. Let Λ^* be the symplectic dual lattice of Λ, which is equal to the sublattice $\{\sum_l n_l f_l \in \Lambda_0^* \mid n_{i_1} + n_{i_2} + n_{i_3} \in 2\mathbb{Z}\}$ of Λ_0^*.

[7]For a unitrivalent graph, the vector space generated by admissible weights serves as a combinatorial realization of the space of conformal blocks for marked surface. But we do not treat such cases in this article.

Note that Λ_0 is a sublattice of Λ with $\Lambda/\Lambda_0 \cong (\mathbb{Z}/2)^{2g-3}$. This follows from the facts $E_1^i \equiv E_2^i \equiv E_3^i \bmod \Lambda_0$ for $i = 1, \cdots, 2g-2$ and $\sum_i E_1^i \in \Lambda_0$.

Now fix an embedding of Γ into $\mathbb{R}^3$ and let B_Γ be a regular neighborhood of Γ in $\mathbb{R}^3$. Then $C_\Gamma := \partial B_\Gamma$ is a closed oriented surface of genus g. Note that C_Γ is endowed with a corresponding pants decomposition associated with Γ. Then we can realize $H_1(C_\Gamma; \mathbb{Z}/2)$ as follows.

Proposition 3.5. *For an embedded trivalent graph $\Gamma \subset \mathbb{R}^3$ and the closed oriented surface C_Γ we have the following.*

(1) *There is a canonical isomorphism $\Lambda^*/2\Lambda_0^* \cong H_1(\Gamma; \mathbb{Z}/2) (\cong (\mathbb{Z}/2)^g)$.*
(2) *There is a canonical exact sequence*

$$0 \to \Lambda_0/2\Lambda \to H_1(C_\Gamma; \mathbb{Z}/2) \to \Lambda^*/2\Lambda_0^* \to 0,$$

and if we fix a section $\Gamma \hookrightarrow C_\Gamma \subset B_\Gamma$ then it induces a splitting of this exact sequence.

Proof. 1. The required isomorphism is given as follows. By definition each $\lambda \in \Lambda^*/2\Lambda_0^*$ has two nontrivial entries at each vertex, and by putting edges corresponding to nontrivial entries in λ together we obtain a cycle in Γ. Conversely for a given cycle in Γ we have an element in $\Lambda^*/2\Lambda_0^*$ whose nontrivial entries correspond with edges on the cycle. (See Fig. 5.)

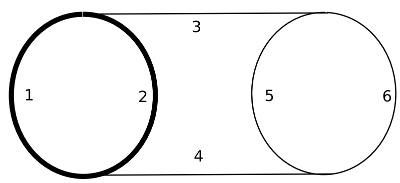

FIGURE 5. The thick line corresponds to $\lambda = f_1 + f_2$

2. The map $H_1(C_\Gamma; \mathbb{Z}/2) \to \Lambda^*/2\Lambda_0^*$ is induced by the inclusion $C_\Gamma \hookrightarrow B_\Gamma$ and the isomorphism in 1. This map is surjective and whose kernel is generated by meridians. On the other hand there are natural homology classes in $H_1(C_\Gamma)$ associated with boundary curves of pairs of pants ($\leftrightarrow \{e_l\}$), and they generate the subgroup consisting of meridians in $H_1(C_\Gamma)$. Since elements in 2Λ correspond with boundaries of pairs of pants (up to orientations) in C_Γ we have a homomorphism $\Lambda_0/2\Lambda \to H_1(C_\Gamma; \mathbb{Z}/2)$. The image of this homomorphism is isomorphic to the subgroup generated by meridians. Note that $\Lambda_0/2\Lambda$ is isomorphic to $(\mathbb{Z}/2)^g$ because there is a natural surjection $\Lambda/2\Lambda (\cong (\mathbb{Z}/2)^{3g-3}) \to \Lambda_0/2\Lambda$ and $\Lambda/\Lambda_0 \cong (\mathbb{Z}/2)^{2g-3}$. So we have an isomorphism between $\Lambda_0/2\Lambda$ and the subgroup of meridians. In this way we have the exact sequence

$$0 \to \Lambda_0/2\Lambda \to H_1(C_\Gamma; \mathbb{Z}/2) \to \Lambda^*/2\Lambda_0^* \to 0.$$

If we fix a section $\Gamma \hookrightarrow C_\Gamma \subset B_\Gamma$, then we have an injection $\Lambda^*/2\Lambda_0^* \hookrightarrow H_1(C_\Gamma; \mathbb{Z}/2)$ and it induces a splitting of this exact sequence. □

Remark 3.1. If we fix a section $\Gamma \hookrightarrow C_\Gamma \subset B_\Gamma$ then by the construction one can check that the isomorphism $H_1(C_\Gamma; \mathbb{Z}/2) \cong \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$ is compatible with their symplectic structures. Namely, the mod 2 intersection pairing between meridians and longitudes coincides with the natural pairing between $\Lambda_0/2\Lambda$ and

$\Lambda^*/2\Lambda_0^*$ induced from the pairing between Λ_0 and Λ_0^*. We denote this pairing by $\cdot : \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* \to \mathbb{Z}/2,\ (\mu, \lambda) \mapsto \mu \cdot \lambda$.

Remark 3.2. In [8] Yoshida carried out an abelianization of $SU(2)$ Wess-Zumino-Witten model and constructed an explicit basis of the vector space $H^0(R_g; \mathcal{L}^{\otimes k})$ parameterized by $QCG_k(\Gamma)$. He considered *Prym varieties* associated with two-fold branched coverings branching at two points in each pair of pants. The lattices Λ_0 and Λ_0^* are used to describe the period lattice of Prym varieties.

4. Construction of the Heisenberg action

In this section we first define actions of $\Lambda_0/2\Lambda$ and $\Lambda^*/2\Lambda_0^*$ on $\mathbb{C}(\Gamma; k)$. By combining them we would like to have a Heisenberg action whose character coincides with that in [1] and [5]. We see that the character formula of the $\Lambda_0/2\Lambda$-action coincides. On the other hand the character formula of the naive $\Lambda^*/2\Lambda_0^*$-action does not coincide. One needs a modification of the naive action. We introduce a notion of the *external edge condition* in Definition 4.6, which gives a condition to give a required modification. In Subsection 4.3 we define a Heisenberg group as a central extension of $\Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$ and the action of the group on $\mathbb{C}(\Gamma; k)$. The main theorem in this section is Theorem 4.16 which asserts that under the external edge condition our Heisenberg action is isomorphic to that in [1] and [5].

4.1. Action of $\Lambda_0/2\Lambda$.

We define an action of $\Lambda_0/2\Lambda$ on the vector space $\mathbb{C}(\Gamma; k)$, which corresponds to the action of A-cycles on the space of theta functions in the $U(1)$-case.

Definition 4.1. We define an action of $\Lambda_0/2\Lambda$ on $\mathbb{C}(\Gamma; k)$ by $\mu : |j\rangle \mapsto (-1)^{2j_\mu}|j\rangle$ for $\mu \in \Lambda_0/2\Lambda$, where we put $j_\mu := \sum_{l, \mu_l \neq 0} j_l \bmod \mathbb{Z}$. This does not depend on a choice of lifts of μ to Λ_0 because of the integrality condition, $j_{i_1} + j_{i_2} + j_{i_3} \in \mathbb{Z}$, in the QCG_k-condition.

Remark 4.1. This action is extracted from the action of $\Lambda_0/2\Lambda$ on theta functions on Prym varieties in [8].

In Theorem 4.11 we show that the character of this action coincides with that in [1] and [5].

4.2. Action of $\Lambda^*/2\Lambda_0^*$.

In this subsection we define an action of $\Lambda^*/2\Lambda_0^*$ on $\mathbb{C}(\Gamma; k)$, which corresponds to the action of B-cycles on the space of theta functions in the $U(1)$-case. Strictly speaking we define an action for each *twisted 1-cocycle* explained later.

Definition 4.2. We can define an action of $\Lambda^*/2\Lambda_0^*$ on $QCG_k(\Gamma)$ as follows;

$$\lambda : j \mapsto \lambda \cdot j = (j_1, \cdots, \frac{k}{2} - j_l, \cdots, j_{3g-3}), \text{for all } l \text{ with } \lambda_l \neq 0.$$

Indeed one can check that this operation preserves the QCG_k-condition by direct computations.

Example 4.3. For the graph Γ in Fig. 5 and $\lambda = f_1 + f_2$, the action of λ on $QCG_k(\Gamma)$ is

$$\lambda : (j_1, j_2, j_3, j_4, j_5, j_6) \mapsto \left(\frac{k}{2} - j_1, \frac{k}{2} - j_2, j_3, j_4, j_5, j_6\right).$$

Remark 4.2. This action is extracted from the action of $\Lambda^*/2\Lambda_0^*$ on theta-characteristics of theta functions on Prym varieties constructed in [8].

The action of $\Lambda^*/2\Lambda_0^*$ on QCG_k induces an action of $\Lambda^*/2\Lambda_0^*$ on $\mathbb{C}(\Gamma;k)$; $\rho_1 : \Lambda^*/2\Lambda_0^* \to GL(\mathbb{C}(\Gamma;k))$, $\rho_1(\lambda)|j\rangle := |\lambda \cdot j\rangle$. By definition the trace of $\rho_1(\lambda)$ is equal to the number of the elements in $QCG_k^\lambda(\Gamma)$, where $QCG_k^\lambda(\Gamma) := \{j \in QCG_k \mid \lambda \cdot j = j\}$. But as the following example shows, one has ${}^\# QCG_k^\lambda(\Gamma) \neq \frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1}$ for $\lambda \neq 0$ in general.

Example 4.4. Let Γ be the graph in Fig. 5 and $\lambda = f_1 + f_2$. Then for $k = 2$ one has ${}^\# QCG_k^\lambda(\Gamma) = 8$ and $\left(\frac{k+2}{2}\right)^{g-1} = 4$.

Take a map $\gamma : \Lambda^*/2\Lambda_0^* \to (\mathbb{C}^\times)^{QCG_k(\Gamma)}$, $\gamma(\lambda) = (\gamma_j(\lambda))$ satisfying the condition

$$\gamma(\lambda_1 + \lambda_2) = \gamma(\lambda_1)\left(\lambda_1 \cdot \gamma(\lambda_2)\right),$$

where we consider the action of $\Lambda^*/2\Lambda_0^*$ on $(\mathbb{C}^\times)^{QCG_k(\Gamma)}$ defined by the interchange of entries, $\lambda : c = (c_j) \mapsto \lambda \cdot c = (c_{\lambda \cdot j})$. Here $(\cdot)^{QCG_k(\Gamma)}$ is the direct product indexed by the finite set $QCG_k(\Gamma)$. Such γ is a twisted 1-cocycle of $\Lambda^*/2\Lambda_0^*$, and it induces an action of $\Lambda^*/2\Lambda_0^*$ on $\mathbb{C}(\Gamma;k)$; $\rho_\gamma : \Lambda^*/2\Lambda_0^* \to GL(\mathbb{C}(\Gamma;k))$, $\rho_\gamma(\lambda)|j\rangle := \gamma_j(\lambda)|\lambda \cdot j\rangle$. By definition the trace of $\rho_\gamma(\lambda)$ is equal to $\sum_{j \in QCG_k^\lambda(\Gamma)} \gamma_j(\lambda)$. To reconstruct the action in [1] and [5], we have to find a twisted 1-cocycle $\gamma : \Lambda^*/2\Lambda_0^* \to (\mathbb{C}^\times)^{QCG_k(\Gamma)}$ satisfying

$$\sum_{j \in QCG_k^\lambda(\Gamma)} \gamma_j(\lambda) = \frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1}$$

for $0 \neq \lambda \in \Lambda^*/2\Lambda_0^*$. Now we consider $\mathbb{Z}/2 (\subset \mathbb{C}^\times)$-valued twisted 1-cocycles and give a condition for cocycles to have a required modification of the action. Here we regard $\mathbb{Z}/2 = \{0, 1\}$ and describe in an additive way. To describe the condition we introduce two notions, λ-*external edges* and λ-*internal edges*.

Definition 4.5 (λ-external edge and λ-internal edge)**.** For $\lambda \in \Lambda^*/2\Lambda_0^*$ an edge $f_l \in \Gamma$ is said to be a λ-*external edge* if the cycle λ on Γ does not pass through f_l and one of the vertex of f_l lies on λ and the other is not. If λ does not pass through f_l and all vertices of f_l lie on λ, then we call f_l is a λ-*internal edge*. See Fig. 6. For $\lambda \in \Lambda^*/2\Lambda_0^*$ we denote the set of all λ-external (resp. λ-internal) edges by $Ex(\lambda)$ (resp. $In(\lambda)$).

By using the notion of external edges we introduce a notion of the *external edge condition* for a map from $\Lambda^*/2\Lambda_0^*$ to $(\mathbb{Z}/2)^{QCG_k(\Gamma)}$. (The internal edges are used in the proof of Proposition 4.13.)

Definition 4.6 (External edge condition)**.** Let $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$, $\delta(\lambda) = (\delta_j(\lambda))$ be a map. We say that δ satisfies the *external edge condition* if the following condition (Ex) is satisfied;

$$\text{(Ex)} \quad \delta_j(\lambda) = \sum_{f_l \in Ex(\lambda)} j_l \bmod 2 \quad \text{for } j \in QCG_k^\lambda(\Gamma).$$

We see in Theorem 4.11 that the external edge condition (Ex) gives a sufficient condition to have a required modification. This means that if a twisted 1-cocycle δ satisfies the external edge condition (Ex), then the character formula of the induced representation $\rho_\delta : \Lambda^*/2\Lambda_0^* \to GL(\mathbb{C}(\Gamma;k))$ coincides with that in [1] and [5].

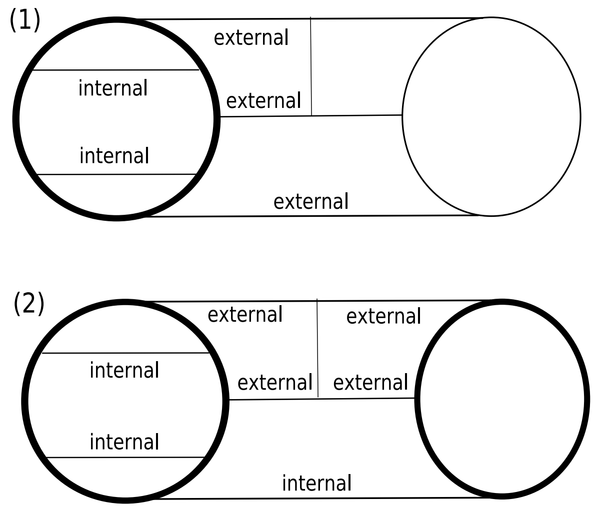

FIGURE 6. Thick lines correspond to $\lambda \in \Lambda^*/2\Lambda_0^*$ in the figures.

Remark 4.3. A twisted 1-cocycle δ satisfying the external edge condition (Ex) is unique up to coboundaries by the following proposition.

Proposition 4.7. *Let G be an abelian group acting on a finite set A and R an abelian group. Then G naturally acts on R^A by interchanging the entries; $g : (r_a)_{a\in A} \mapsto (r_{g\cdot a})_{a\in A}$ $(g \in G)$. Let $\tau : G \to R^A$, $\tau(g) = (\tau_a(g))$ be a twisted 1-cocycle. Then the cohomology class of τ is equal to 0 in $H^1(G; R^A)$ if and only if $\tau_a(g) = 0$ for all $a \in A$ with $g \cdot a = a$.*

Proof. First we assume that $\tau_a(g) = 0$ for $a \in A$ with $g \cdot a = a$. Fix representatives $\{a_1, \cdots, a_N\}$ of A/G and put $r_{g\cdot a_k} := \tau_{a_k}(g)$ for $k = 1, \cdots, N$ and $g \in G$. The well-definedness of this definition follows from the following fact. For $g_1, g_2 \in G$ such that $g_1 \cdot a = g_2 \cdot a$, we have $\tau_a(g_1) = \tau_a(g_2)$ because of the equalities

$$\begin{aligned} 0 = \tau_a(g_1 - g_2) &= \tau_{g_1\cdot a}(-g_2) + \tau_a(g_1) \\ &= \tau_{g_2\cdot a}(-g_2) + \tau_a(g_1) \\ &= -\tau_a(g_2) + \tau_a(g_1). \end{aligned}$$

Then by definition $r = (r_{g\cdot a_k})$ satisfies $\tau_a(g) = r_{g\cdot a} - r_a$ and this means that the cohomology class $[\tau] \in H^1(G; R^A)$ is equal to 0. It is easy to check the converse is true. □

4.3. **Actions of the Heisenberg groups.** In this subsection we define a Heisenberg group as a central extension of $\Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$. By combining the two actions in the previous subsections we define an action of the Heisenberg group on $\mathbb{C}(\Gamma; k)$. In Proposition 4.15 we show that our Heisenberg group gives a reduction to a $\mathbb{Z}/2$-extension as noted by Andersen-Masbaum [1, p.297]. In Theorem 4.11 we compute the character of our Heisenberg action under the external edge condition by using the factorization property and the Verlinde formula. Our computation implies the main result in this subsection (Theorem 4.15), which asserts that our Heisenberg action is isomorphic to that in [1] and [5] (Theorem 4.15).

Definition 4.8. Put $\mathcal{G}(\Gamma) := \mathbb{C}^\times \times (\Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*)$ as a set. We define a group structure on $\mathcal{G}(\Gamma)$ by

$$(c_1, \mu_1, \lambda_1) \cdot (c_2, \mu, \lambda_2) = (c_1 c_2 (-1)^{\lambda_2 \cdot \mu_1}, \mu_1 + \mu_2, \lambda_1 + \lambda_2),$$

where $\cdot$ is the natural pairing between $\Lambda_0/2\Lambda$ and $\Lambda^*/2\Lambda_0^*$ induced from the symplectic form on $\mathbb{R}\langle\{e_l\}\rangle \oplus \mathbb{R}\langle\{f_l\}\rangle$. Then $\mathcal{G}(\Gamma)$ defines a central extension

$$1 \to \mathbb{C}^\times \to \mathcal{G}(\Gamma) \to \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* \to 0$$

whose commutator is given by $(-1)^{\omega_2(\cdot,\cdot)}$, where ω_2 is the mod 2 symplectic form on $H_1(C_\Gamma; \mathbb{Z}/2) = \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$;

$$\omega_2((\mu_1, \lambda_1), (\mu_2, \lambda_2)) := \mu_1 \cdot \lambda_2 + \lambda_1 \cdot \mu_2.$$

Then one can check the following proposition.

Proposition 4.9. *For a twisted 1-cocycle $\gamma : \Lambda^*/2\Lambda_0^* \to (\mathbb{C}^\times)^{QCG_k(\Gamma)}$, the map $\rho_\gamma^{(k)} : \mathcal{G}(\Gamma) \to GL(\mathbb{C}(\Gamma; k))$, $\rho_\gamma^{(k)}(c, \mu, \lambda)|j\rangle = c^k(-1)^{2j_\mu}\gamma_j(\lambda)|\lambda \cdot j\rangle$ is a homomorphism.*

Proof. We prove this proposition by direct computations. For $(c_i, \mu_i, \lambda_i) \in \mathcal{G}(\Gamma)$ $(i = 1, 2)$ we have

$$\begin{aligned} & \rho_\gamma^{(k)}(c_1, \mu_1, \lambda_1)\rho_\gamma^{(k)}(c_2, \mu_2, \lambda_2)|j\rangle \\ = \ & \rho_\gamma^{(k)}(c_1, \mu_1, \lambda_1)\left(c_2^k(-1)^{2j_{\mu_2}}\gamma_j(\lambda_2)|\lambda_2 \cdot j\rangle\right) \\ = \ & (c_1c_2)^k(-1)^{2((\lambda_2\cdot j)_{\mu_1} + j_{\mu_2})}\gamma_{\lambda_2\cdot j}(\lambda_1)\gamma_j(\lambda_2)|\lambda_1 \cdot \lambda_2 \cdot j\rangle \\ = \ & (c_1c_2)^k(-1)^{k\lambda_2\cdot\mu_1}(-1)^{2(j_{\mu_1} + j_{\mu_2})}\gamma_j(\lambda_1 + \lambda_2)|(\lambda_1 + \lambda_2) \cdot j\rangle \\ = \ & \rho_\gamma^{(k)}\left((c_1, \mu_1, \lambda_1) \cdot (c_2, \mu_2, \lambda_2)\right)|j\rangle. \end{aligned}$$

Note that in the third equal sign we used the relation

$$(\lambda_2 \cdot j)_{\mu_1} = \frac{k}{2}\lambda_2 \cdot \mu_1 + j_{\mu_1} \bmod \mathbb{Z}.$$

□

By definition for an element c in the center of $\mathcal{E}(\Gamma)$, the action of $\rho_\gamma^{(k)}(c, 0, 0)$ on $\mathbb{C}(\Gamma; k)$ is given by the multiplication of c^k and the trace of $\rho_\gamma^{(k)}(1, \mu, \lambda)$ is given by

$$Tr(\rho_\gamma^{(k)}(1, \mu, \lambda)) = \sum_{j \in QCG_k^\lambda(\Gamma)} (-1)^{2j_\mu}\gamma_j(\lambda).$$

Lemma 4.10. *If k is an odd number, then for any twisted 1-cocycle γ we have $Tr(\rho_\gamma^{(k)}(1, \mu, \lambda)) = 0$ $((\mu, \lambda) \neq (0, 0))$.*

Proof. First note that if $\lambda \cdot j = j$ then $j_l = \frac{k}{4}$ for all l with $\lambda_l \neq 0$. But $\frac{k}{4}$ is not half integer if k is odd. This implies that for an odd number k and $\lambda \neq 0$ we have $QCG_k^\lambda(\Gamma) = \emptyset$, and hence $Tr(\rho_\gamma^{(k)}(1, \mu, \lambda); \mathbb{C}(\Gamma; k)) = 0$. Now we assume that k is an odd number and $\lambda = 0$, $\mu \neq 0$. In this case we have

$$Tr(\rho_\gamma^{(k)}(1, \mu, 0); \mathbb{C}(\Gamma; k)) =^\# \{j \in QCG_k \mid j_\mu \in \mathbb{Z}\} -^\# \{j \in QCG_k \mid j_\mu \notin \mathbb{Z}\}$$

by definition. Take and fix a cycle $\lambda_\mu \in \Lambda^*/2\Lambda_0^*$ on Γ with $\mu \cdot \lambda_\mu \equiv 1 \bmod 2$. Then a map

$$\begin{array}{rcl} \{j \in QCG_k \mid j_\mu \in \mathbb{Z}\} & \longleftrightarrow & \{j \in QCG_k \mid j_\mu \notin \mathbb{Z}\} \\ j & \longleftrightarrow & \lambda_\mu \cdot j \end{array}$$

gives a bijection if k is an odd number because of the relation

$$(\lambda_\mu \cdot j)_\mu = \frac{k}{2}\lambda_\mu \cdot \mu + j_\mu \bmod \mathbb{Z},$$

in particular we obtain $Tr(\rho_\gamma^{(k)}(1,\mu,0);\mathbb{C}(\Gamma;k)) = 0$. □

We have the following trace formula for a twisted 1-cocycle $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ satisfying the external edge condition (Ex).

Theorem 4.11. *If a twisted 1-cocycle $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ satisfies the external edge condition (Ex), then we have*

$$Tr(\rho_\delta^{(k)}(1,\mu,\lambda);\mathbb{C}(\Gamma;k)) = (-1)^{\frac{k}{2}\mu\cdot\lambda}\frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1}$$

for $(\mu,\lambda)(\neq(0,0)) \in \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$.

Now we make several preparations to show the theorem. It is enough to consider the case when k is even due to Lemma 4.10. By the assumption (δ satisfies the external edge condition), the trace of $\rho_\delta^{(k)}$ is given by

$$\begin{aligned}
\sum_{j\in QCG_k^\lambda(\Gamma)} (-1)^{2j_\mu+\delta_j(\lambda)} \quad = \quad & {}^\#\{j \in QCG_k^\lambda \mid 2j_\mu + \sum_{f_l\in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
& -{}^\#\{j \in QCG_k^\lambda \mid 2j_\mu + \sum_{f_l\in Ex(\lambda)} j_l \equiv 1 \bmod 2\} \\
= \quad & {}^\#\{j \in QCG_k^\lambda \mid 2j_{\mu(\lambda)} + \frac{k}{2}\mu\cdot\lambda + \sum_{f_l\in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
& -{}^\#\{j \in QCG_k^\lambda \mid 2j_{\mu(\lambda)} + \frac{k}{2}\mu\cdot\lambda + \sum_{f_l\in Ex(\lambda)} j_l \equiv 1 \bmod 2\} \\
= \quad & (-1)^{\frac{k}{2}\mu\cdot\lambda}\Bigg({}^\#\{j \in QCG_k^\lambda \mid 2j_{\mu(\lambda)} + \sum_{f_l\in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
& -{}^\#\{j \in QCG_k^\lambda \mid 2j_{\mu(\lambda)} + \sum_{f_l\in Ex(\lambda)} j_l \equiv 1 \bmod 2\}\Bigg),
\end{aligned}$$

where we take and fix lifts of μ and λ, and $\mu(\lambda)$ is defined by $\mu(\lambda) := \sum_l \epsilon_l(1-\epsilon_l')e_l$ for $\mu = \sum_l \epsilon_l e_l$ and $\lambda = \sum_l \epsilon_l' f_l$. See the following example.

Example 4.12. Let Γ be a graph in Fig.7. If we take $\mu = e_1 + e_7 + e_8$ and

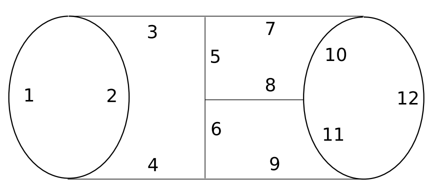

FIGURE 7.

$\lambda = f_1 + f_2 + f_{10} + f_{11} + f_{12}$, then one has $\mu(\lambda) = e_7 + e_8$.

We compute the number in the last parenthesis. For given $\lambda \in \Lambda^*/2\Lambda_0^*$ we construct graphs $\Gamma(\lambda)$ and $\Gamma'(\lambda)$ from Γ as follows. (See also Fig. 8.)

$$\Gamma(\lambda) := \{f_l, v_i \mid f_l \text{ with } \lambda_l \neq 0 \text{ or } f_l \in Ex(\lambda) \cup In(\lambda), v_i \text{ is a vertex of } f_l\},$$
$$\Gamma'(\lambda) := \{f_l, v_i \mid f_l \text{ with } \lambda_l = 0 \text{ or } f_l \in Ex(\lambda), v_i \text{ is a vertex of } f_l\}.$$

We assume that $\Gamma(\lambda)$ has $3g_1-3+2n$ edges and $\Gamma'(\lambda)$ has $3g_2-3+2n$ edges (then $g = g_1 + g_2 + n - 1$), and $Ex(\lambda) = \{f_1, \cdots, f_n\}$. Put $QCG_k^\lambda(\Gamma(\lambda); j_1', \cdots, j_n') := \{j \in QCG_k(\Gamma(\lambda); j_1', \cdots, j_n') \mid \lambda \cdot j = j\} = \{j \in QCG_k(\Gamma(\lambda); j_1', \cdots, j_n') \mid j_l = \frac{k}{4} \text{ if } \lambda_l \neq 0\}$.

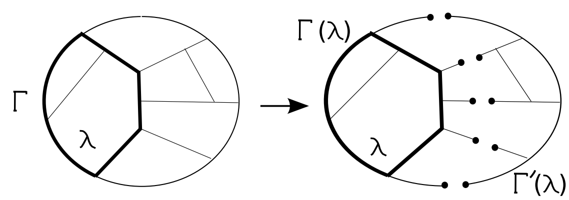

FIGURE 8.

Proposition 4.13. *We have the following;*

$$^\# QCG_k^\lambda(\Gamma(\lambda); j_1', \cdots, j_n') = \begin{cases} \left(\frac{k+2}{2}\right)^{g_1-1} & (j_1', \cdots, j_n' \in \{0, 1, \cdots, k/2\} \subset \mathbb{Z}) \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. First note that a triple of half integers $(\frac{k}{4}, \frac{k}{4}, j_l)$ satisfies the QCG_k-condition if and only if $j_l \in \{0, 1, \cdots, \frac{k}{2}\} \subset \mathbb{Z}$, and hence one has that $QCG_k^\lambda(\Gamma(\lambda); j_1', \cdots, j_n') \neq \emptyset$ only if $j_1', \cdots, j_n' \in \{0, 1, \cdots, k/2\} \subset \mathbb{Z}$. Next we assume that this condition is satisfied. By the construction there are $g_1 - 1$ edges in $In(\lambda) \subset \Gamma(\lambda)$. On the other hand elements in $QCG_k^\lambda(\Gamma(\lambda); j_1', \cdots, j_n')$ have weight $\frac{k}{4}$ at edges on λ and integer weights $0, \cdots, \frac{k}{2}$ at edges in $In(\lambda)$. In addition such integer weights at edges in $In(\lambda)$ can be taken arbitrary. This implies that there are $\left(\frac{k+2}{2}\right)^{g_1-1}$ elements in $QCG_k^\lambda(\Gamma(\lambda); j_1', \cdots, j_n')$. □

Now we assume that there are m edges with $\mu(\lambda)_l \neq 0$ which has no univalent vertices in $\Gamma'(\lambda)$. We have the following.

Proposition 4.14.

$$\sum_{j_1', \cdots, j_n'} (-1)^{j_1'+\cdots+j_n'} \Big(^\#\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid 2j_{\mu(\lambda)} \equiv 0 \bmod 2\}$$
$$- ^\#\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid 2j_{\mu(\lambda)} \equiv 1 \bmod 2\}\Big)$$
$$= \left(\frac{k+2}{2}\right)^{g_2-1+n}.$$

Proof. Cut edges in $\Gamma'(\lambda)$ at f_l with $\mu(\lambda)_l \neq 0$, and denote the graph obtained in this way by $\Gamma'(\lambda)_\mu$. See Fig. 9. Then by using the factorization property and the

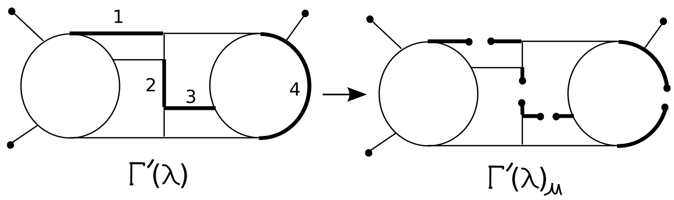

FIGURE 9. $\Gamma'(\lambda)_\mu$ for $\mu(\lambda) = e_1 + e_2 + e_3 + e_4$.

Verlinde formula the number in the first parenthesis can be computed as follows.

$$
\begin{aligned}
&{}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j'_1, \cdots, j'_n) \mid 2j_{\mu(\lambda)} \equiv 0 \bmod 2\} \\
&\qquad\qquad - {}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j'_1, \cdots, j'_n) \mid 2j_{\mu(\lambda)} \equiv 1 \bmod 2\} \\
= &\sum_{j''_1, \cdots, j''_n} (-1)^{2(j''_1 + \cdots + j''_n)} {}^{\#}QCG_k(\Gamma'(\lambda)_\mu; j'_1, \cdots, j'_n, j''_1, j''_1, \cdots, j''_m, j''_m) \\
= &\sum_{j''_1, \cdots, j''_n} (-1)^{2(j''_1 + \cdots + j''_n)} \left(\frac{k+2}{2}\right)^{g_2 - m - 1} \sum_{l=1}^{k+1} \frac{\prod_{n'=1}^n \sin \frac{2j'_{n'}+1}{k+2} l\pi \prod_{m'=1}^m \sin^2 \frac{2j''_{m'}+1}{k+2} l\pi}{\left(\sin \frac{l\pi}{k+2}\right)^{2g_2 - 2 + n}} \\
= &\left(\frac{k+2}{2}\right)^{g_2 - m - 1} \sum_{l=1}^{k+1} \frac{\prod_{n'=1}^n \sin \frac{2j'_{n'}+1}{k+2} l\pi \sum_{j''_1, \cdots, j''_n} (-1)^{2(j''_1 + \cdots + j''_m)} \prod_{m'=1}^m \sin^2 \frac{2j''_m+1}{k+2} l\pi}{\left(\sin \frac{l\pi}{k+2}\right)^{2g_2 + n - 2}} \\
= &\left(\frac{k+2}{2}\right)^{g_2 - 1} \prod_{n'=1}^n \sin \frac{2j'_{n'}+1}{2} \pi.
\end{aligned}
$$

Here we used the following elementary equalities in the last equal sign;

$$
\begin{aligned}
&\sum_{j''_1, \cdots, j''_m} (-1)^{2(j''_1 + \cdots + j''_m)} \prod_{m'=1}^m \sin^2 \frac{2j''_{m'}+1}{k+2} l\pi \\
= &\left(\sum_{a=0}^{k/2} \left(\sin^2 \frac{2a+1}{k+2} l\pi - \sin^2 \frac{2a+2}{k+2} l\pi \right) \right)^m \\
= &\left(\frac{1}{2} \sum_{a=0}^{k/2} \left(-\cos 2\frac{2a+1}{k+2} l\pi + \cos 2\frac{2a+2}{k+2} l\pi \right) \right)^m = \begin{cases} \left(\frac{k+2}{2}\right)^m & (l = \frac{k+2}{2}) \\ 0 \ (l \neq \frac{k+2}{2}). & \end{cases}
\end{aligned}
$$

Then the number in the proposition can be computed as follows.

$$
\begin{aligned}
&\left(\frac{k+2}{2}\right)^{g_2 - 1} \sum_{j'_1, \cdots, j'_n} (-1)^{j'_1 + \cdots + j'_n} \prod_{m=1}^n \left(\sin \frac{2j'_m + 1}{2} \pi \right) \\
= &\left(\frac{k+2}{2}\right)^{g_2 - 1} \left(\sum_{a=0}^{\frac{k}{2}} (-1)^a \sin \frac{2a+1}{2} \pi \right)^n = \left(\frac{k+2}{2}\right)^{g_2 - 1 + n}.
\end{aligned}
$$

□

By using Proposition 4.13 and Proposition 4.14 we can complete the proof of Theorem 4.11 as follows.

Proof of Theorem 4.11. By previous results we have

$$
\begin{aligned}
&{}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} + \sum_{f_l \in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
&\qquad\qquad\qquad -{}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} + \sum_{f_l \in Ex(\lambda)} j_l \equiv 1 \bmod 2\} \\
&= {}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} \equiv 0, \sum_{f_l \in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
&\qquad\qquad\qquad -{}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} \equiv 0, \sum_{f_l \in Ex(\lambda)} j_l \equiv 1 \bmod 2\} \\
&+{}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} \equiv 1, \sum_{f_l \in Ex(\lambda)} j_l \equiv 1 \bmod 2\} \\
&\qquad\qquad\qquad -{}^{\#}\{j \in QCG_k^{\lambda} \mid 2j_{\mu(\lambda)} \equiv 1, \sum_{f_l \in Ex(\lambda)} j_l \equiv 0 \bmod 2\} \\
&= \sum_{j_1', \cdots, j_n'} (-1)^{j_1'+\cdots+j_n'}\, {}^{\#}QCG_k^{\lambda}(\Gamma(\lambda); j_1', \cdots, j_n')\, {}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid j_{\mu(\lambda)} \in \mathbb{Z}\} \\
&- \sum_{j_1', \cdots, j_n'} (-1)^{j_1'+\cdots+j_n'}\, {}^{\#}QCG_k^{\lambda}(\Gamma(\lambda); j_1', \cdots, j_n')\, {}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid j_{\mu(\lambda)} \notin \mathbb{Z}\} \\
&= \left(\frac{k+2}{2}\right)^{g_1-1} \sum_{j_1', \cdots, j_n'} (-1)^{j_1'+\cdots+j_n'} \big({}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid j_{\mu(\lambda)} \in \mathbb{Z}\} \\
&\qquad\qquad\qquad\qquad -{}^{\#}\{j \in QCG_k(\Gamma'(\lambda); j_1', \cdots, j_n') \mid j_{\mu(\lambda)} \notin \mathbb{Z}\}\big) \\
&= \left(\frac{k+2}{2}\right)^{g_1-1} \left(\frac{k+2}{2}\right)^{g_2+n-1} = \left(\frac{k+2}{2}\right)^{g-1}.
\end{aligned}
$$

Here we used the factorization property in the second equal sign, Proposition 4.13 in the third equal sign and Proposition 4.14 in the fourth equal sign. □

Since the 2-cocycle of the $\mathbb{C}^{\times}$-central extension $\mathcal{G}(\Gamma)$ takes values in $\mathbb{Z}/2$, it can be reduced to the $\mathbb{Z}/2$-central extension $\mathcal{E}'(\Gamma)$;

$$\mathbb{Z}/2 \to \mathcal{E}'(\Gamma) \to \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*,$$

and it induces the $\mathbb{Z}/4$-central extension $\mathcal{E}(\Gamma)$;

$$\mathbb{Z}/4 \to \mathcal{E}(\Gamma) \to \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$$

via the natural injection $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$. In [1, p.297] Andersen-Masbaum noted that the $\mathbb{Z}/4$-central extension $\mathcal{E}(J^{(2)}; \mathcal{L})$ of $H^1(C_\Gamma; \mathbb{Z}/2) \cong H_1(C_\Gamma; \mathbb{Z}/2)$ can be reduced to a $\mathbb{Z}/2$-central extension. Our construction gives such a reduction. Namely;

Proposition 4.15. *The $\mathbb{Z}/4$-central extension $\mathcal{E}(\Gamma)$ is isomorphic to the central extension $\mathcal{E}(J^{(2)}; \mathcal{L})$ in* [1].

Proof. By definition the 2-cocycle τ_1 of $\mathcal{E}(\Gamma)$ is given by

$$\tau_1((\mu_1, \lambda_1), (\mu_2, \lambda_2)) = (-1)^{\lambda_2 \cdot \mu_1} \quad ((\mu_i, \lambda_i) \in \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*).$$

Fix a set-theoretical section $s : H_1(C_\Gamma;\mathbb{Z}/2) \to H_1(C_\Gamma;\mathbb{Z}/4)$ of the natural surjection $H_1(C_\Gamma;\mathbb{Z}/4) \twoheadrightarrow H_1(C_\Gamma;\mathbb{Z}/2)$. By Theorem 2.1 the 2-cocycle τ_2 of $\mathcal{E}(J^{(2)};\mathcal{L})$ associated with the section s is given by

$$\tau_2(a,b) = \sqrt{-1}^{\omega_4(s(a),s(b))} \quad (a,b \in H_1(C_\Gamma;\mathbb{Z}/2)).$$

Since τ_1 takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$ we can take a map $q : \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* \to \mathbb{Z}/4$ satisfying $q(a)^2 = \tau_1(a,a)$ and $q(a+b) = q(a)q(b)\left(\tau_1(a,b)\tau_1(b,a)\right)^{\text{"}\frac{1}{2}\text{"}}$ for $a,b \in H_1(C_\Gamma;\mathbb{Z}/2)$. Then one can check the relation

$$\tau_1(a,b)\tau_2(a,b)^{-1} = q(a)q(a+b)^{-1}q(b) \quad (a,b \in H_1(C_\Gamma;\mathbb{Z}/2)),$$

and the map

$$\mathcal{E}(\Gamma) \ni (c,a) \mapsto cq(a)\rho_{s(a)} \in \mathcal{E}(J^{(2)};\mathcal{L}) \quad (a \in H_1(C_\Gamma;\mathbb{Z}/2))$$

gives an isomorphism. □

For each twisted 1-cocycle $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ there are induced representations of $\mathcal{E}'(\Gamma)$ and $\mathcal{E}(\Gamma)$ via $\rho_\delta^{(k)}$. We denote these representations by the same letter $\rho_\delta^{(k)}$. If the twisted 1-cocycle satisfies the external edge condition (Ex), then we have the character formula Theorem 4.11. The the following is the main theorem in this section.

Theorem 4.16. *If a twisted 1-cocycle δ satisfies the external edge condition* (Ex), *then the representation $\rho_\delta^{(k)}$ of $\mathcal{E}(\Gamma)$ is isomorphic to the representation of $\mathcal{E}(J^{(2)};\mathcal{L})$ on the space of holomorphic sections $H^0(R_g;\mathcal{L}^{\otimes k})$.*

Proof. This theorem follows from Theorem 4.11 and the isomorphism given in Proposition 4.15. For $(\mu,\lambda) \in \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* = H_1(C_\Gamma;\mathbb{Z}/2)$ we have

$$\begin{aligned} Tr(cq(\mu,\lambda)\rho_{s(\mu,\lambda)}; H^0(R_g;\mathcal{L}^{\otimes k}) &= c^k q(\mu,\lambda)^k Tr(\rho_{s(\mu,\lambda)}; H^0(R_g;\mathcal{L}^{\otimes k}) \\ &= c^k(-1)^{\frac{k}{2}\lambda\cdot\mu}\frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1} \\ &= Tr(\rho_\delta(c,\mu,\lambda); \mathbb{C}(\Gamma;k)), \end{aligned}$$

where we used Theorem 2.2 and the fact $q(\mu,\lambda) = \tau_1((\mu,\lambda),(\mu,\lambda))^{\frac{1}{2}} = (-1)^{\frac{1}{2}\lambda\cdot\mu}$ in the second equal sign. This completes the proof of the theorem. □

5. Explicit constructions

In this section we give explicit constructions of the twisted 1-cocycles $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ satisfying the external edge condition (Ex). We construct the cocycles for all planar trivalent graphs in Subsection 5.1, and for some non-planar trivalent graphs in Subsection 5.2. The existence of such cocysles is shown in our paper [6]. Moreover we have a finite algorithm to construct such cocycles in the paper. Constructions given in this section are examples given by the algorithm.

The proof of Lemma 4.10 implies that if k is odd then the trivial 1-cocycle $\delta \equiv 0$ satisfies the external edge condition automatically. Hence constructions we need are only for the case when k is even.

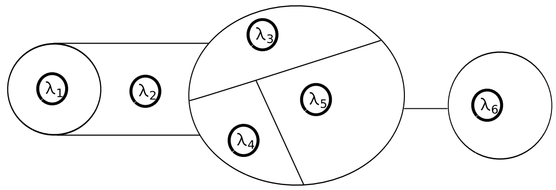

FIGURE 10.

5.1. **planar case.** Let Γ be a planar trivalent graph and fix an embedding of Γ into $\mathbb{R}^2$. Then $\mathbb{R}^2$ is decomposed into $g+1$ domains by edges of Γ, and one finds the canonical basis $\lambda_1, \cdots, \lambda_g$ of $H_1(\Gamma;\mathbb{Z}/2)$ as boundaries of the domains. (See also Fig. 10.)

For $h = 1, \cdots, g$, we define $\delta(\lambda_h) = (\delta_j(\lambda_h)) \in (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ by

$$\delta_j(\lambda_h) := \sum_{f_l \in Ex(\lambda_h)} j_l + \sum_{f_{l'} \in In(\lambda_h)} 2j_{l'} \quad (j \in QCG_k(\Gamma)).$$

Lemma 5.1. *For $h, h' = 1, \cdots, g$, we have $\delta(\lambda_h)+\lambda_h\cdot\delta(\lambda_{h'}) = \lambda_{h'}\cdot\delta(\lambda_h)+\delta_j(\lambda_{h'})$.*

Proof. If two cycles λ_h and $\lambda_{h'}$ in Γ are disjoint, then we have $\lambda_{h'}\cdot\delta(\lambda_h) = \delta(\lambda_h)$ and $\lambda_h\cdot\delta(\lambda_{h'}) = \delta(\lambda_{h'})$ by definition, and hence the relation in the lemma holds in this case. Consider the case $\lambda_h \cap \lambda_{h'} \neq \emptyset$. In this case the following three cases in Fig. 11 occur.

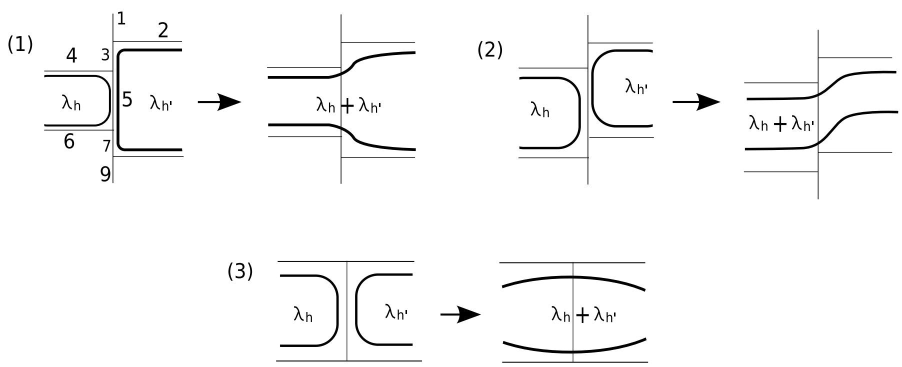

FIGURE 11.

In the case of (1) we have

$$\begin{aligned}\delta_j(\lambda_h) + \delta_{\lambda_h\cdot j}(\lambda_{h'}) &= (j_2+j_6) + (j_1 - j_3 - j_5 + j_7) \\ &= (-j_2 - j_6) + (j_1 + j_3 + j_5 + j_7) = \delta_{\lambda_{h'}\cdot j}(\lambda_h) + \delta_j(\lambda_{h'}).\end{aligned}$$

Here we used the fact which follows form the QCG_k-condition, $j_2+j_6-(j_3+j_5) \in \mathbb{Z}$. By quite similar computations one can check that the relation holds in the cases of (2) and (3). $\square$

By this lemma the definition $\delta(\lambda_h + \lambda_{h'}) := \delta(\lambda_h) + \lambda_h \cdot \delta(\lambda_{h'}) \in (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ makes sense for all $h, h' = 1, \cdots, g$. Moreover we can define a map $\delta = (\delta_j) : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ in an inductive way. Namely if $\delta(\sum_{\lambda\in A}\lambda) \in (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ is defined for a subset $A \subset \{\lambda_1, \cdots, \lambda_g\}$, then we can define $\delta(\lambda_h + \sum_{\lambda\in A}\lambda)$ by

$\delta(\lambda_h + \sum_{\lambda \in A} \lambda) := \delta(\lambda_h) + \lambda_h \cdot \delta(\sum_{\lambda \in A} \lambda)$ for all $h = 1, \cdots, g$. One can check the well-definedness of this definition by induction on ${}^{\#}A$. By definition this δ is a twisted 1-cocycle.

Theorem 5.2. *The twisted 1-cocycle* $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ *defined above satisfies the external edge condition* (Ex).

Proof. By definition $\delta(\lambda_1), \cdots, \delta(\lambda_g)$ satisfy the condition (Ex). Assume that $\delta(\sum_{h \in A} \lambda_h)$ satisfies the condition (Ex) for a subset $A \subset \{1, \cdots, g\}$. Then it is enough to check that $\delta(\lambda + \sum_{h \in A} \lambda_h) = \lambda \cdot \delta(\sum_{h \in A}) + \delta(\lambda)$ satisfies the condition (Ex) for $\lambda = \lambda_1, \cdots, \lambda_g$. Essentially it is enough to check for five cases in Fig. 12. We denote $\sum_{h \in A} \lambda$ by λ' in the figure.

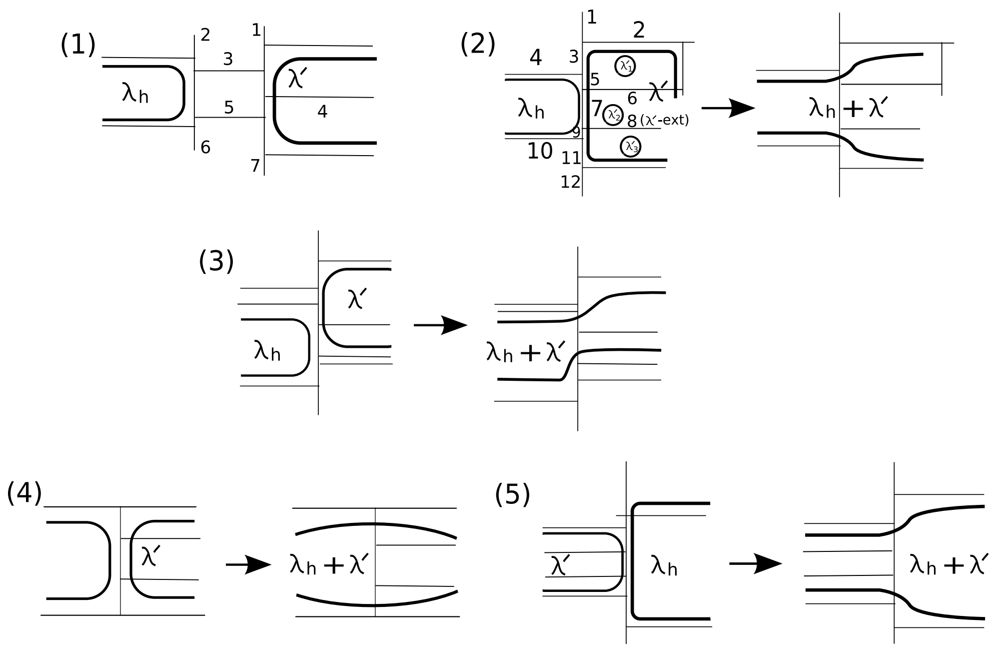

FIGURE 12.

We first remark that $\delta_j(\lambda')$ depends only on the weights of edges in λ', $Ex(\lambda')$ and $In(\lambda')$. In the case of (1), the condition (Ex) holds because f_3 and f_5 are λ_h-external and λ'-external, and they become $(\lambda_h + \lambda')$-internal edges. Next we consider the case of (2). For each $j \in QCG_k(\Gamma)$ we can compute $\delta_j(\lambda')$ as $\delta_{(\lambda'_3+\lambda'_2)\cdot j}(\lambda'_1) + \delta_{\lambda'_3 \cdot j}(\lambda'_2) + \delta_j(\lambda'_3)$, and hence $\delta_j(\lambda')$ contains $j_1 + j_4 + j_5 \pm j_8 - j_9 + j_{10} + j_{12}$. (Note that j_7 contributes to $\delta(\lambda')$ by $j_7 - j_7 = 0$.) Since $\delta_j(\lambda_h)$ contains $j_3 + j_6 + j_8 + j_{11}$, the sum $\delta_{\lambda_h \cdot j}(\lambda') + \delta_j(\lambda_h)$ contains $j_1 + j_3 - j_4 - j_5 + j_6 + j_8 \pm j_8 + j_9 - j_{10} + j_{11} + j_{12}$, and if $\lambda \cdot j = j$ then $j_3 = j_4 = j_{10} = j_{11} = \frac{k}{4}$, and $\delta_j(\lambda_h + \lambda') = \delta_{\lambda_h \cdot j}(\lambda') + \delta_j(\lambda_h)$ contains $j_1 + \frac{k}{4} - \frac{k}{4} - j_5 + j_6 + (j_8 \pm j_8) + j_9 - \frac{k}{4} + \frac{k}{4} + j_{12} = j_1 - j_5 + j_6 + (j_8 \pm j_8) + j_9 + j_{12}$. Note that if $j_3 = j_4 = \frac{k}{4}$ then one has $j_5 \in \mathbb{Z}$ by the QCG_k-condition and hence $-j_5 \equiv j_5 \bmod 2$. By the similar argument we can see that $j_8 \in \mathbb{Z}$ and hence $j_8 \pm j_8 \equiv 0 \bmod 2$. Summarizing, $\delta(\lambda_h + \lambda')$ contains $j_1 + j_5 + j_6 + j_9 + j_{12}$ and this implies that $\delta(\lambda_h + \lambda')$ satisfies the condition (Ex). By quite similar arguments one

can see that $\delta(\lambda_h + \lambda')$ satisfies the condition (Ex) in the cases of (3), (4) and (5), and hence we complete the proof by induction. $\square$

Example 5.3. Here we give a typical example. Consider the graph Γ in Fig. 13 and the canonical basis $\lambda_1, \cdots, \lambda_g$. For $j \in QCG_k(\Gamma)$, put

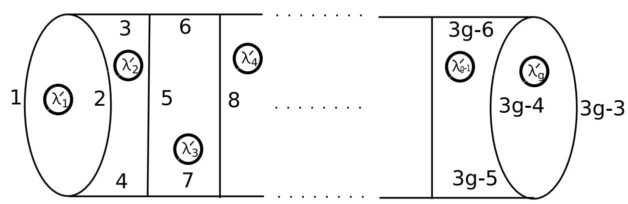

FIGURE 13.

$$\begin{cases} \delta_j(\lambda_1) = j_3 + j_4 \\ \delta_j(\lambda_2) = 2j_1 + j_6 + j_7 \\ \delta_j(\lambda_3) = j_3 + j_4 + j_9 + j_{10} \\ \cdots \\ \delta_j(\lambda_{g-1}) = j_{3g-9} + j_{3g-8} + 2j_{3g-3} \\ \delta_j(\lambda_g) = j_{3g-6} + j_{3g-5}, \end{cases}$$

then one can check that $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ is a twisted 1-cocycle satisfying the external edge condition (Ex).

5.2. **Non-planar examples.** We can construct examples of the twisted 1-cocycles $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ satisfying the external edge condition (Ex) for non-planar graphs as follows.

Example 5.4. Consider the trivalent graph Γ in Fig. 14. One can check that this graph is non-planar by reduction to absurdity and computations of Euler numbers.

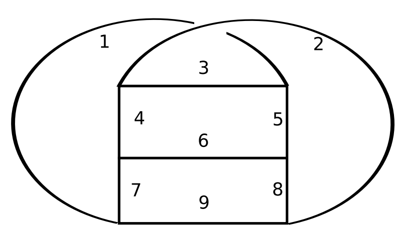

FIGURE 14.

Take $\lambda_1 := f_3 + f_4 + f_5 + f_6$, $\lambda_2 := f_6 + f_7 + f_8 + f_9$, $\lambda_3 := f_1 + f_5 + f_8 + f_9$ and $\lambda_4 := f_2 + f_4 + f_7 + f_9$ as a basis of $\Lambda^*/2\Lambda_0^* = H_1(\Gamma; \mathbb{Z}/2)$. For $j \in QCG_k(\Gamma)$, put

$$\begin{aligned} \delta(\lambda_1) &:= j_1 + j_2 + j_7 - j_8 \\ \delta(\lambda_2) &:= -j_1 - j_2 + j_4 + j_5 \\ \delta(\lambda_3) &:= j_2 + j_3 + j_6 + j_7 + j_8 + j_9 \\ \delta(\lambda_4) &:= j_1 + j_3 + j_6 + j_7 + j_8 - j_9. \end{aligned}$$

By direct computations one can see that these define a twisted 1-cocycle $\delta : \Lambda^*/2\Lambda_0^* \to (\mathbb{Z}/2)^{QCG_k(\Gamma)}$ and if $k \equiv 0 \bmod 4$ then it satisfies the external edge condition (Ex).

By adding edges to this graph one can construct examples of δ for non-planar graphs for all $g \geq 4$. For example, add two vertices on f_1 and connect them by new edge as in Fig. 15, and change $\delta(\lambda_3)$ in Example 14 as follows.

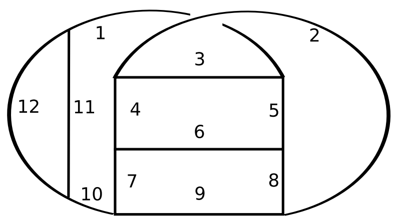

Figure 15.

Take $\lambda_3' := f_{11}+f_{12}$, $\lambda_3 := f_1+f_5+f_8+f_9+f_{10}+f_{12}$ and $\lambda_1, \lambda_2, \lambda_4$ in Example 14 as a basis of $\Lambda^*/2\Lambda_0^*$. Then $\delta(\lambda_3') := j_1 + j_{10}$ and $\delta(\lambda_1), \cdots, \delta(\lambda_4)$ define a twisted 1-cocycle satisfying the external edge condition (Ex) if $k \equiv 0 \bmod 4$.

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